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DOCTRINAL FUNCTIONS.

THE term propositional function, invented by Mr. Bertrand Russell, is perhaps the weightiest that has entered the literature of logic and mathematics in the course of a hundred years. It has the rare distinction of being a perfect name for a supreme concept. I am not about to expound its meaning at length nor to attempt to justify my estimate of its significance. It seems desirable, however, to remind the reader of so much of the term's meaning as will be essential to an understanding of the principal thesis of this paper.

Let it be recalled, then, that a propositional function is any statement containing one or more variables. If we denote these by x, y, z, *etc.*, then such simple statements as x is a philosopher, $x = 2$, x is a brother of y, $3x + 2y = 5$, x has been divinely appointed by y to subjugate z, $4x - 3y + 9z = 7$, will serve to exemplify what is meant by a propositional function. It is of fundamental importance to bear in mind that propositional functions, though they have the form of propositions, are not propositions. A proposition is a statement that is true or else false, but a propositional function is neither true nor false. The statements, $2 + 5 = 7$, $3 + 6 = 7$, are propositions, one of them true, the other one false; but the statement, $x + y = 7$, is neither true nor false; it is not a proposition but is a propositional function.

To derive propositions from propositional functions it is evidently necessary to substitute for the variables present in the latter what we may call constants, or terms of definite meaning; but such substitution, though necessary, is not sufficient, for it is always possible to select such constants as will, if substituted for the variables of a given function, convert the latter, not into a proposition, but into non-sense, it being understood that a non-sensical statement is one involving a contradiction in terms. Suppose, for example, that our given function is the statement, x is an integer less than 5. The *class* of all integers less than 5 is a constant, a definite somewhat. Substituting this constant for the variable x, we get the statement, the class of all integers less than 5 is an integer less than 5. This is neither a propositional function nor a proposition; it is non-sensical, the non-sense, or contradiction in terms, consisting in talking of a class of things as if a given class could conceivably be one of the things composing it.

The constants that, when substituted for the variables in a given propositional function, convert it into non-sense, may be called *inadmissible* constants for that function; all other constants, since they convert the function into propositions, true or false, may be called the *admissible* constants for the function. The admissible constants for a given function fall into two classes: those that convert it into true propositions and those that convert it into false ones. It is convenient to call the constants of the former class *verifiers* of the function; and those of the latter class *falsifiers* of the function. The *verifiers of a function* are said to *satisfy* it and are called the *values* of its *variables;* and the propositions derived from a function by substitution of values of its variables for the variables themselves are called *values* of the *function.* Thus the values of a given function are the true propositions that are derivable from it by replacing its variables by admissible constants.

These ideas and distinctions being premised, I turn to the task of this paper, which is that of viewing in a somewhat new light the essential nature of what has come to be known in mathematics and modern logic as the postulational method of founding and constructing mathematical sciences or branches of mathematical science. What I shall say of postulate systems in general the reader may most readily grasp and verify by having constant reference to some specific system with which he may chance to be specially familiar, as, for example, the widely known system for Euclidean metric geometry, presented by Hilbert in his Foundations of Geometry.

It is well known that any postulate system (or system of assumptions or of axioms or of primitives as the postulates are variously called) contains one or more undefined terms, and that at least one of these denotes an *element,* that is to say, a thing or a substantive as distinguished from a relation. In the Hilbert system, for example, there are three undefined element names, point, line, and plane; in Veblen's System of Axioms for Geometry[1] there is one such name, point; and the same is true of Pierí's beautiful system for elementary geometry, of which an excellent account is found in the late Louis Couturat's *Les Principes des Mathématiques,* p. 192.

What is the office of the undefined element names in a postulate system? Undoubtedly the presence of these lends the system the appearance of having a definite subject-matter. The appearance, however, is appearance only: a postulate system has no definite subject-matter. We are said to interpret such a system—to assign it a definite subject-matter—when we assign to the undefined terms any specific significance reconcilable with the postulates. How many interpretations does a postulate system admit? At a later stage of this paper it will be proved that any postulate system admits of any given transfinite cardinal number of interpretations. At present it is sufficient to bear in mind the familiar fact that a postulate system admits more than one interpretation, for this fact shows that the rôle of the undefined terms is the rôle of variables, and it is this aspect of the matter to which I wish to draw attention. Because of the presence of variables in its statements a postulate system is not a system of propositions, as it is commonly said to be, but it is a system of propositional functions; and the same is, of course, to be said of all statements, or theorems, that are logically deducible from the postulates, for all such theorems involve the same variables as do the postulates themselves.

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[1] *Transactions of the American Mathematical Society,* Vol. 5, p. 343.

Now let P denote any given postulate system, and denote by T the set of any or all theorems deducible from P. Consider the body B of statements comprising both those of P and those of T. What ought B to be called? A doctrine? Certainly not, for a doctrine ought to have a specific subject-matter and it ought to be true or else false; but B, being composed of propositional functions, has no such subject-matter and is neither true nor false. Observe, however, that, though B is no doctrine, it has the *form* of a doctrine; note also that B contains variables and that it is, therefore, itself a variable in the sense in which any function of one or more variables is itself a variable. The suggestion arising from these considerations is inevitable and compelling: B ought to be called a doctrinal function.

A function to be a function must have values. What are the values of a doctrinal function? To answer, consider the facts. In order to interpret the system P, it is necessary and sufficient to replace P's variables by verifiers of its functions. In doing so, however, we pass from the system P (of propositional functions) to a new system P' of propositions, values of the functions, and it is the new system that has a definite subject-matter and never the old one; so, too, it is P''s statements, and never those of P, that are true. Just as P''s propositions are values of P's functions, so the system P' itself ought to be called a value of the system P. In interpreting P, what happens to T? Since the propositional functions T are logical consequences of the system P, it is plain that, in passing by interpretation from P to P', we at the same time pass from the set T of functions to a set T' of propositions, at once values of T's functions and consequences of P''s propositions. Evidently T' ought to be called a value of T. Now, since the act of interpreting P carries us from P to P' and from T to T', it is evident that the same act carries us from B, composed of P and T, to a B', composed of P' and T'; and just as P' and T' are, respectively, values of P and T, so ought B' to be called a value of B. This B, however, is a doctrinal function.

What is the value B'? Being a body of true propositions about a definite subject-matter, it evidently is a doctrine, and the doctrine is true. Accordingly our question as to what the expression, a value of a doctrinal function, ought to mean, is answered: a value of a doctrinal function is a true doctrine derivable from the function by substituting for its variables verifiers of its postulates.

How many values has a doctrinal function? Let P denote, as before, the system of postulates of the function. The postulates contain at least one variable element, e. Let I denote one interpretation of P, giving rise to one doctrine, D, a value of the function. Let e' denote the verifier of the postulates that was used in obtaining I. The verifier e' is a symbol for a class of things e'_1, e'_2, e'_3 $\cdots$ satisfying the relations stated by the postulates. Now let O be any given object of thought. The object O and any given one of the e''s together constitute a pair of things. Denote by ϵ the class of all the pairs thus obtainable, and let the pairs in ϵ be ϵ_1, ϵ_2, ϵ_3, $\cdots$. It is plain that a one-to-one correspondence subsists between the e''s and the ϵ's. We agree to say that any set of the ϵ's satisfy a given relation when the relation is satisfied by the e''s contained in the given ϵ's. Hence instead of e' we may use ϵ as a verifier of P. So doing, we get an interpretation I' and therewith a corresponding doctrine D'. As ϵ is different from e', I' is different (in content or subject-matter) from I, and D' similarly differs from D. Evidently there are thus obtainable as many different doctrines—as many different values of the given doctrinal function—as there are different objects O of thought. We have, therefore, the remarkable proposition that the number of values of any doctrinal function is equal to any given transfinite cardinal number.

A given doctrinal function has infinitely many values not included in the infinitude obtainable in the manner indicated in the preceding paragraph, and it is those not so obtainable that men happen to be most interested in. It will be sufficient to illustrate this twofold fact by means of a familiar example. Let our doctrinal function be that one which consists of the above-mentioned Hilbert system of postulates together with all theorems deducible from them, and let us denote it by DF. By Hilbert the three variable elements are called point, line, and plane. To avoid prejudice these names ought to be replaced by symbols denoting variables, as v_1, v_2, v_3; then the postulate, two distinct points always completely determine a line, would read: two distinct v_1's always completely determine a v_2. And so on for the remaining postulates. A reader will find it very instructive to make the indicated substitutions. Let us suppose it done. Our doctrinal function DF now discourses about the v's and about nothing else. To interpret *DF*, to derive one of its values, to get a true doctrine from it, it is necessary and sufficient to replace the v's by verifiers of the postulates involving them. One set of verifiers is obtained by letting v_1, v_2, v_3 denote, respectively, point, line and plane, allowing these names to have the significance they have had ever since they were described by Euclid in his Elements. The resulting value of *DF* is the familiar doctrine traditionally known as the Euclidean geometry of the point, line, and plane.

It is well known that another value of *DF* is obtainable by replacing the v's as follows: v_1 by a triad (x, y, z) of real numbers; v_2 by the system of triads satisfying a pair of equations, $Ax + By + Cz + D = 0$, $A'x + B'y + C'z + D' = 0$, it being supposed that the equations have a common solution and that in neither of them are all the coefficients zero; by v_3 the system of triads satisfying one such equation.[2] The doctrine thus arising from *DF* is evidently not a geometry; it is absolutely free of all reference to any idea of spatial extension; it is a pure algebra, the algebra of triads of real numbers.

Given the Euclidean geometry of the point, line, and plane, it is possible to find among its configurations three geometric entities (other than point, line and plane) which will serve as verifiers of the Hilbert postulates and will accordingly give us another value of *DF*. This value, like the former, is a geometric doctrine, but it differs from the former in content. In like manner, from this second value of *DF*—from this second doctrine—a third one can be obtained, and so on indefinitely. Proof of this fact is omitted for lack of room. The fact means, however, that among the values of *DF* there is a denumerable infinitude of doctrines that must be called geometric because each of them deals with a subject-matter involving essential reference to spatial extension. It is true also that among the values of *DF* there is a denumerable infinitude of algebraic or numerical doctrines; and it must be added that *DF* owns a denumerable infinitude of values, or doctrines, that in point of content are neither geometric nor numerical. All these doctrines are isomorphic with one another and with their common matrix, the doctrinal function *DF*, from which they spring. The values of a doctrinal function differ from the function in that they have content while it has not, and in the further fact that the values are true while the function is neither true nor false. The doctrines differ from one another in respect to *kind* of content. This last difference is psychological and not logical.

The relation of a doctrinal function to its values casts a clear light on the relation of form to content, of logic to psychology. We happen to live in a world where an infinitude of psychologically diverse doctrines are logically one. How unfortunate it would be to live in a world where no two doctrines were isomorphic!

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[2] A proof that the indicated values of the v's are verifiers of the Hilbert postulates is found in Weber and Wellstein's *Elementare Geometrie,* 2d ed., p. 83.

As an interesting corollary to the foregoing discussion it must be said that Hilbert's Foundations of Geometry is not a geometry at all, nor is it any other doctrine; it is a doctrinal function having an infinitude of values, some of them geometric, some of them algebraic, some of them neither the one nor the other. The reader will perceive that analogous remarks are applicable to any and every other postulationally established scientific structure involving variables.

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ERROR IN PROFESSOR HOLT'S REALISM

THERE is much that as a realist I have learned from Mr. Holt's realistic writings. On very many points I can not agree with him, but even where this is the case, I generally know why; his clear discussion has always helped to clear away difficulties, and if it has not contributed a satisfactory solution, it makes a satisfactory solution easier to obtain.

In this paper I wish to give my reason for thinking that Mr. Holt has not solved the problem of error. What I take to be his failure confirms me in my conviction that a favorite realistic enterprise of these days is very unpromising. Perhaps the majority of contemporary realists insist that there is nothing in consciousness that *in any way* owes its being to its being in consciousness. Everything that is has its own "independent" being, is "a neutral entity." Some neutral entities may be dependent upon *other* neutral entities, but "one salient fact about them, in so far as they enter into individual experiences (though this is really a fact about experience), is that 'their being experienced makes no difference to them,' they remain what they are!"[1] The greatest difficulty which those who do not hold this view find in it is the difficulty of providing a place for error. Many different provisions have been suggested, and it is Mr. Holt's solution of this difficulty that we are to examine now.

Mr. Holt maintains "that all errors are cases of contradiction or contrariety. One has met error who has experienced that *A* is *B* and that the same *A* is not *B*. But the experiencing is not the significant fact, and that all errors are of knowledge is true merely by definition, since contrariety or contradiction is called 'error' only when it occurs in some person's field of consciousness."[2] "Every

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[1] C. C., *i. e.*, Holt's *The Concept of Consciousness*, p. 103.

[2] N. R., *i. e.*, *The New Realism*, p. 361.